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THE FINITE STURM-LIOUVILLE TRANSFORM

By

A. Cemal Eringen

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Department of Mechanics
Illinois Institute of Technology
Technology Center
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A. Cemal Eringen
Illinois Institute of Technology

1. Introduction

It is known that certain partial differential equations can be solved with the use of particular types of definite integrals having appropriate kernels. The choice of kernel depends on the type of boundary value problem. The solution obtained by these transforms is direct in the sense that it contains the boundary values in the solution. This, of course, is lacking in the classical approach. Finite Fourier transforms are of this type [1].

Recently Tranter [2], used a Legendre polynomial as a kernel. Scott [3] following a similar approach used a Jacobi polynomial as a kernel which extends the result of [2].

It is the purpose of the present paper to extend and unify all such special transforms. Thus we employ a kernel which may be determined to suit each particular type of problem. The transform can be employed to solve a wide class of linear second order partial differential equations.

In this paper we deal only with those transforms whose intervals are finite. Thus, the results obtained here are particularly useful for finite domains, or domains which are finite in one direction. An extension to infinite domains and singular cases will be made in a later paper.

2. The Sturm-Liouville Expansion

A second order, linear, homogeneous differential equation containing an arbitrary parameter λ has the general form:

$$Mv \equiv p_0 v_{,yy} + p_1 v_{,y} + (p_2 + \lambda p_3)v = 0 \tag{1}$$

$$a_0 \leq y \leq b_0$$

where $p_i = p_i(y)$; v and y are the dependent and real independent variables, respectively. Indices after comma represent differentiation, i.e.: $\phi_{,y} = d\phi/dy$.

Equation (1) can be transformed into the canonical form:

$$Lu = \lambda u\ , \qquad L \equiv q(x) - \frac{d^2}{dx^2}\ , \qquad a \leq x \leq b \tag{2}$$

where:

$$\left.\begin{aligned} &u = \theta(x)v\ , \qquad x = \int (p_3/p_0)^{1/2} dy \\ &a = \int^{a_0} (p_3/p_0)^{1/2} dy\ , \qquad b = \int^{b_0} (p_3/p_0)^{1/2} dy \\ &q(x) = \frac{\theta_{,xx}(x)}{\theta(x)} - \phi(x)\ , \qquad \theta(x) = (p_3 k^2/p_0)^{1/4} \\ &\phi(x) = p_2/p_3\ , \qquad k = \exp\int (p_1/p_0)\, dy \end{aligned}\right\} \tag{3}$$

It is simpler to work with (2).

We consider two solutions $\phi(x,\lambda)$ and $\chi(x,\lambda)$ of (2) satisfying the boundary conditions:

$$\begin{aligned} &\phi(a,\lambda) = \sin\alpha\ , \qquad \phi_{,x}(a,\lambda) = -\cos\alpha \\ &\chi(b,\lambda) = \sin\beta\ , \qquad \chi_{,x}(b,\lambda) = -\cos\beta \end{aligned} \tag{4}$$

<u>Both solutions are unique if</u> $q(x)$ <u>is real, continuous everywhere in</u> (a,b) <u>and has finite limits for</u> $x = a$ <u>and</u> $x = b$

(see [4], p.6). We can build up $\phi(x,\lambda)$ and $\chi(x,\lambda)$ as a linear combination of two independent solutions $\phi_o(x,\lambda)$ and $\chi_o(x,\lambda)$ of (2). For let:

$$\left.\begin{aligned} \omega_o(\lambda)\phi(x,\lambda) &= \phi_o(x,\lambda)[\chi_o(a,\lambda)\cos\alpha + \chi_{o,x}(a,\lambda)\sin\alpha] \\ &\quad - \chi_o(x,\lambda)[\phi_o(a,\lambda)\cos\alpha + \phi_{o,x}(a,\lambda)\sin\alpha] \\ \omega_o(\lambda)\chi(x,\lambda) &= \phi_o(x,\lambda)[\chi_o(b,\lambda)\cos\beta + \chi_{o,x}(b,\lambda)\sin\beta] \\ &\quad - \chi_o(x,\lambda)[\phi_o(b,\lambda)\cos\beta + \phi_{o,x}(b,\lambda)\sin\beta] \\ \omega_o(\lambda) &= W(\phi_o,\chi_o) = \phi_o(x,\lambda)\chi_{o,x}(x,\lambda) - \phi_{o,x}(x,\lambda)\chi_o(x,\lambda) \end{aligned}\right\} \quad (5)$$

Functions $\phi(x,\lambda)$ and $\chi(x,\lambda)$ defined by (5) satisfy (2) and (4). Here $W(u,v)$ is the Wronskian, and <u>can be shown to be independent of</u> x.

Let λ_n be a root of $W(\phi,\chi) = 0$. Then we have:

$$\omega(\lambda_n) = W(\phi_n,\chi_n) = \phi_n\chi_{n,x} - \phi_{n,x}\chi_n = 0$$

or

$$\phi_{n,x}/\phi_n = \chi_{n,x}/\chi_n$$

where:

$$\phi_n = \phi(x,\lambda_n), \qquad \text{and} \quad \chi_n = \chi(x,\lambda_n).$$

Hence the integration gives:

$$\chi(x,\lambda_n) = k_n\phi(x,\lambda_n) \tag{6}$$

Consequently $\phi(x,\lambda_n)$ and $k_n\phi(x,\lambda_n)$ <u>satisfy all of the boundary conditions</u> (4).

It is also easy to show that $\phi(x,\lambda_n)$ <u>is an orthogonal set</u>. For, in view of (2), (4) and (6) we have:

$$(\lambda_m-\lambda_n)\int_a^b \phi_m\phi_n dx = \int_a^b (\phi_m L\phi_n - \phi_n L\phi_m) dx = W(\phi_m, \phi_n) = 0$$

$$(\lambda_m \neq \lambda_n)$$

Hence we can write:

$$\int_a^b \phi(x,\lambda_n)\phi(x,\lambda_m) dx = N^2\delta_{mn} \tag{7}$$

where δ_{mn} is the Kronecker delta (equal to zero for $m\neq n$, one for $m=n$)

It can be shown also that <u>all roots</u> of $\omega(\lambda) = 0$ <u>are real and distinct</u> (see [4], pp 11-12).

From (5) and (6) it follows that:

$$k_n = \frac{\phi_o(b,\lambda_n)\cos\beta + \phi_{o,x}(b,\lambda_n)\sin\beta}{\phi_o(a,\lambda_n)\cos\alpha + \phi_{o,x}(a,\lambda_n)\sin\alpha}$$

$$= \frac{\chi_o(b,\lambda_n)\cos\beta + \chi_{o,x}(b,\lambda_n)\sin\beta}{\chi_o(a,\lambda_n)\cos\alpha + \chi_{o,x}(a,\lambda_n)\sin\alpha} \tag{8}$$

<u>If now</u> $f(x)$ <u>is an integrable function over</u> (a,b) <u>and if</u> $a<x<b$, <u>we have the Sturm-Liouville expansion of</u> $f(x)$:

$$f(x) = \sum_{n=o}^{\infty} [k_n/\omega_{,\lambda}(\lambda_n)]\phi(x,\lambda_n)\int_a^b \phi(y,\lambda_n) f(y) dy \tag{9}$$

For the convergence of (9) to $f(x)$ see [4] pp 12-15 or [5] pp275-276.

3. <u>Finite Sturm-Liouville Transform</u>

<u>Let</u> $f(x)$ <u>be a real continuous and integrable function of</u> x, <u>in the interval</u> (a,b). <u>We define the finite Sturm-Liouville transform</u> $S\{f\} \equiv F(\lambda_n)$ associated with system (2) and (4) by:

$$F(\lambda_n) \equiv S\{f\} = \int_a^b f(y)\Psi(y,\lambda_n) dy \tag{10}$$

where:

$$\psi(y,\lambda_n) = [k_n/\omega_{,\lambda}(\lambda_n)]^{1/2}\phi(y,\lambda_n) \tag{11}$$

The inversion theorem now follows from (9):

$$f(x) = \sum_{n=o}^{\infty} \mathcal{F}(\lambda_n)\psi(x,\lambda_n) \tag{12}$$

Here functions $\psi(y,\lambda_n)$ are orthonormal, i.e.:

$$\int_a^b \psi(x,\lambda_n)\psi(x,\lambda_m)dx = \delta_{mn} \tag{13}$$

This is readily seen from (7) and (11).

Equations (10) and (12) are basic for the solution of some partial differential equations.

4. Transform of Lf

We are now going to prove that:

$$S_x\left\{Lf\right\} \equiv S_x\left\{q(x)f - \partial^2 f/\partial x^2\right\} = B_f(\lambda_n,t) + \lambda_n \mathcal{F}(\lambda_n,t) \tag{14}$$

where:

$$B_f(\lambda_n,t) = [k_n/\omega_{,\lambda}(\lambda_n)]^{1/2}\left\{-f(b,t)\frac{\cos\beta}{k_n} - f_{,x}(b,t)\frac{\sin\beta}{k_n} + f(a,t)\cos\alpha + f_{,x}(a,t)\sin\alpha\right\} \tag{15}$$

is a function depending on the boundary values of $f(x,t)$, $f_{,x}(x,t)$ at $x = a,b$. Subscript f implies that the function B contains boundary values of the function f. Here S_x represents the finite Sturm-Liouville transform taken with respect to x:

$$S_x\left\{ \phi(x,t) \right\} = \bar{\phi}(\lambda_n, t) \tag{16}$$

To prove (14) we mutliply Lf by $\psi(x,\lambda_n)$ and integrate between (a,b), that is, by definition we have:

$$S_x\left\{ Lf \right\} = \int_a^b [q(x)f(x,t) - \partial^2 f/\partial x^2]\psi(x,\lambda_n)\,dx \tag{17}$$

Integrating the second term in the integrand of (17) twice by part we obtain:

$$S_x\left\{ Lf \right\} = -[f_{,x}\,\psi - \psi_{,x} f]_a^b + \int_o^b f(x,t)L\psi(x,\lambda_n)\,dx \tag{18}$$

If we now use (4) and (6) in the first term of the right hand side and (2) inside the integrand we obtain (14).

5. Solution of Some Partial Differential Equations

a) Let us consider a parabolic partial differential equation

$$Mv + p_4 v_{,t} = 0 \tag{19}$$

where Mv is defined by (1), $p_i = p_i(y,t)$ are known and $v(y,t)$ is the unknown dependent variable.

The use of transformation (3) reduces (19) to:

$$u_{,xx} + [\lambda - p(x,t)]u + (p_4/p_3)u_{,t} = 0 \tag{20}$$

where:

$$p(x,t) = q(x,t) - p_4 p_o (k/2p_o)^2 \frac{\partial}{\partial t}(p_o/p_3\ k^2) \tag{21}$$

Here $q(x,t)$, $k(t)$ and $\lambda(t)$ are given by (3) with t as a parameter. If $\phi(x,t,\lambda_n)$ and $\psi(x,t,\lambda_n)$ are now obtained as before except with $q(x)$ replaced by $p(x,t)$, t a parameter, and p_4/p_3 a function of t alone, then (20), with the use of the finite Sturm-Liouville transform, can be reduced to an ordinary differential equation in t. For, let $p_4/p_3 = r(t)$, then application of the transform to (20) gives:

$$-B_u(\lambda_n,t) + (\lambda-\lambda_n)\bar{u}(\lambda_n,t) + r(t)\bar{u}_{,t}(\lambda_n,t) = 0 \qquad (22)$$

Equation (22) is of first order. The complete solution is:

$$\bar{u}(\lambda_n,t) = C(\lambda_n)\exp\left(\int^t \frac{\lambda_n - \lambda}{r}\,dt\right) + \left[\exp\left(\int^t \frac{\lambda_n - \lambda}{r}\,dt\right)\right] \cdot$$

$$\int^t (B_u/r)\left[\exp\left(-\int^t \frac{\lambda_n - \lambda}{r}\,dt\right)\right]dt \qquad (23)$$

Now, inversion theorem (12) gives:

$$u(x,t) = \sum_{n=o}^{\infty} \bar{u}(\lambda_n,t)\psi_n(x,t,\lambda_n) \qquad (24)$$

Therefore, the solution of (19) is effected. We must, however, remember that at least one of each pair of boundary values $u(a,t)$, $u_{,x}(a,t)$ and $u(b,t)$, $u_{,x}(b,t)$ must be known. By selecting α and β properly in $B_u(\lambda_n,t)$ we can make the terms containing the unknown pair zero. Of course any knowledge of the boundary values of u leading to the complete evaluation of B_u is sufficient for the solution.

b) A second order linear partial differential equation containing a parameter λ has the general form:

$$\begin{aligned} Nv + a_4 v_{,z} + a_5 v_{,\tau} + (a_6 + \lambda a_7)v &= 0 \\ Nv = a_1 v_{,zz} + 2a_2 v_{,z\tau} &+ a_3 v_{,\tau\tau} \end{aligned} \qquad (25)$$

where $a_i = a_i(z,\tau)$. If the equation is of <u>elliptic</u> <u>type</u> we have:

$$a_1 a_3 - a_2^2 > 0 \tag{26}$$

Consider now an elliptic equation (25). Let $\xi(z,\tau) =$ const. and $\eta(z,\tau) =$ const., respectively be the solutions of the following differential equations:

$$d\tau/dz = \zeta_1(z,\tau) \qquad d\tau/dz = \zeta_2(z,\tau)$$

$$\left.\begin{matrix}\zeta_1\\ \zeta_2\end{matrix}\right\} = -\frac{a_2}{a_1} \pm i\left[\frac{a_3}{a_1} - \left(\frac{a_2}{a_1}\right)^2\right]^{1/2} \tag{27}$$

If we set:

$$2y = \xi + \eta \quad , \quad 2it = \xi - \eta \tag{28}$$

equation (25) can be transformed into the canonical form [6]:

$$v_{,yy} + p_1 v_{,y} + (p_2 + \lambda p_3)v + p_4 v_{,t} + v_{,tt} = 0 \tag{29}$$

where:

$$p_1 = \frac{2}{\beta} Py \quad , \; p_2 = 2a_6/\beta \; , \; p_3 = 2a_7/\beta$$

$$p_4 = \frac{2}{\beta} Pt \quad , \; \beta = a_1(y_{,z}^{\;2} + t_{,z}^{\;2}) + 2a_2(y_{,z}y_{,\tau} + t_{,z}t_{,\tau})$$

$$+ a_3(y_{,\tau}^{\;2} + t_{,\tau}^{\;2}) \tag{30}$$

Here operator P is defined by:

$$P\phi = N\phi + a_4\phi_{,z} + a_5\phi_{,\tau} \tag{31}$$

and $N\phi$ is given by the second of (25).

Using the transformation (3) with $p_o = 1$, (29) becomes:

$$u_{,xx} + (\lambda - p)u + r_1 u_{,t} + r_o u_{,tt} = 0 \tag{32}$$

where:

$$p = q(x,t) - mp_4 g_{,t} - mg_{,tt}\ , \quad r_o = p_3^{-1}$$
$$r_1 = mp_4 g + 2mg_{,t}\ , \quad g = (p_3 k^2)^{-1/4},$$
$$m = (k^2/p_3{}^3)^{1/4} \tag{33}$$

In the special case where r_1, and r_o are functions of t alone equation (32) can be transformed into an ordinary differential equation by applying the finite Sturm-Liouville transform. Hence:

$$r_o \overline{u}_{,tt} + r_1 \overline{u}_{,t} + (\lambda - \lambda_n)\overline{u}(\lambda_n, t) = B_u(\lambda_n, t) \tag{34}$$

We can now either use transformation (3) with respect to t, with r_o, r_1 replacing p_o, p_1 and $p_2 = 0$, $p_3 = 1$ and find the solution of (34) and then invert it or we can solve (34) directly for $\overline{u}(\lambda_n, t)$ and then (24) gives $u(x,t)$.

It must be remembered that this method has the advantage that it contains the boundary values of the function within the solution, consequently, the difficulty of satisfying boundary conditions is eliminated. The method is also valid in the cases where the usual technique of separation of variables fail.

6. Solutions ϕ_o and χ_o of Some Special Equations

Below are given two independent solutions ϕ_o and χ_o of some special differential equations (D. E.) of mathematical physics. Their canonical forms (2) will be given by only determining u, q, and λ.

a) Bessel Functions

$$v_{,yy} + \frac{1}{y} v_{,y} + (s^2 - \frac{\nu^2}{y^2})v = 0 \qquad \text{(D. E.)}$$

$$u = x^{1/2}v \quad , \quad y = x \quad , \quad q = (\nu^2 - \tfrac{1}{4})x^{-2}, \quad \lambda = s^2$$

$$\phi_o(x,\lambda) = x^{1/2} J_\nu(xs) \quad , \quad \chi_o(x,\lambda) = x^{1/2}Y_\nu(xs)$$

Interval (a,b) should not contain $x = 0$.

b) Spherical Harmonics

$$[(1-y^2)v_{,y}]_{,y} + [\nu(\nu+1) - \frac{\mu^2}{1-y^2}]v = 0 \quad \text{(D. E.)}$$

$$u = v \cos^{1/2}x \quad , \quad y = \sin x \quad , \quad q = \frac{\mu^2}{\cos^2 x} - \frac{1}{4}\tan^2 x - \frac{1}{2} ,$$

$$\lambda = \nu(\nu+1) \quad , \quad \phi_o(x,\lambda) = \cos^{1/2} x \cdot P_\nu^\mu(\sin x),$$

$$\chi_o(x,\lambda) = \cos^{1/2} x \cdot Q_\nu^\mu(\sin x)$$

Interval (a,b) should not contain (-1, 1).

c) Hermite Polynomials

$$v_{,yy} - yv_{,y} + nv = 0 \qquad \text{(D. E.)}$$

$$u = e^{-x^2/4}v, \quad y = x \quad , \quad q = -\frac{1}{2} + \frac{x^2}{4} \quad , \quad \lambda = n \quad ,$$

$$\phi_o(x,\lambda) = e^{-x^2/4}\, He_n(x) \quad , \quad \chi_o(x,\lambda) = e^{-x^2/4}\, he_n(x)$$

d) Tschebyscheff Polynomials

$$(1-y^2)v_{,yy} - yv_{,y} + n^2v = 0 \qquad \text{(D. E.)}$$

$$u = v \quad , \quad y = \sin x, \quad q = 0 \quad , \quad \lambda = n^2$$

$$\phi_o(x,\lambda) = T_n(\sin x) = \cos n(\sin^{-1}y) \quad ,$$

$$\chi_o(x,\lambda) = U_n(\sin x) = \sin n(\sin^{-1}y)$$

e) <u>Mathieu Functions</u>

$$u_{,xx} + (\lambda - 2h^2\cos 2x)u = 0 \qquad \text{(D. E.)}$$

$$\phi_o(x,\lambda) = ce_{2n}(x), \quad ce_{2n+1}(x) \quad , se_{2n}(x), \quad s_{2n+1}(x)$$

$$\chi_o(x,\lambda) = ce_{2n}{}^{(2)}(x), \quad ce_{2n+1}{}^{(2)}(x), \; se_{2n}^{(2)}(x)$$

$$s_{2n+1}{}^{(2)}(x)$$

($n = 0, 1, 2, \ldots$ except for se_{2n} only $n = 1, 2, \ldots$)

$\phi_o(x,\lambda)$ is periodic.

f) <u>Wittaker Functions</u>

$$v_{,yy} + \left(-\frac{1}{4} + \frac{k}{y} + \frac{\frac{1}{4} - \mu^2}{y^2}\right)v = 0 \qquad \text{(D. E.)}$$

$$u = (x/2)^{1/2}v, \quad y = x^2/4, \quad q = \frac{16\mu^2 - 17}{4x^2} + \frac{x^2}{16}$$

$$\phi_o(x,\lambda) = (x/2)^{1/2}\, W_{k,\mu}(x^2/4), \quad \chi_o(x,\lambda) = (x/2)^{1/2}$$

$$W_{-k,\mu}(-x^2/4) \qquad (*)$$

Interval (a,b) should not contain $x = 0$.

7. <u>Application</u>

As an illustration we solve a heat conduction problem which may be looked upon as a mathematical model of <u>volcanos</u>. The problem also has application in the exhaust parts of jet engines. As far as I know the problem has not previously been solved.

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* Here comma does not mean differentiation.

Consider the conical sheel enclosed by two coaxial cones having the same apex O, and two concentric spheres having O as their center (Fig. 1).

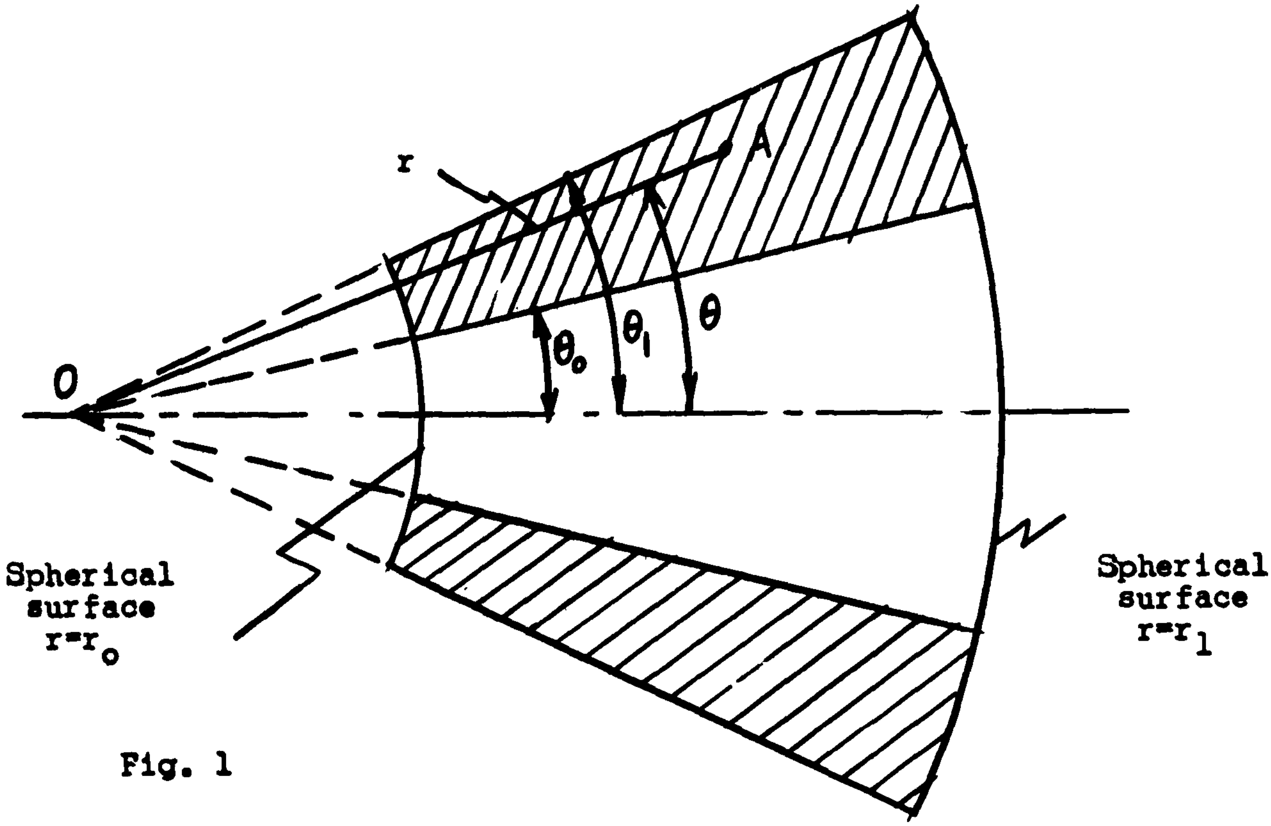

Fig. 1

In polar coordinates the inner and outer conical surfaces are given by $\theta = \theta_o$ and $\theta = \theta_1$, and the end surfaces by $r = r_o$ and $r = r_1$.

Problem: Determine the steady temperature distribution within the shell under the general axi-symmetric boundary conditions in temperature. This is the general Dirichlet problem for the domain under consideration. The differential equation and boundary conditions (B. C.) are given below:

$$(\text{D. E.}) \quad \Delta V = r^{-2}(r^2 V_{,r})_{,r} + r^{-2}[(1-y^2)V_{,y}]_{,y} = 0$$
$$y = \cos\theta \quad , \; r_o < r < r_1 \quad , \; \theta_o < \theta < \theta_1 \qquad (35)$$

$$(\text{B. C.}) \quad \left.\begin{array}{ll} V = V_o(r) & \text{for } \theta = \theta_o \\ V = V_1(r) & \text{for } \theta = \theta_1 \end{array}\right\} r_o < r < r_1$$
$$\left.\begin{array}{ll} V = V_3(\theta) & \text{for } r = r_o \\ V = V_4(\theta) & \text{for } r = r_1 \end{array}\right\} \theta_o < \theta < \theta_1 \qquad (36)$$

Here $V(r,y)$ is the temperature function, r and θ are the polar coordinates.

<u>Solution</u>: We can exclude the second term of (D. E.) (35) if we use a finite Sturm-Liouville transform associated with the Legendre equation:

$$[(1-y^2)v_{,y}]_{,y} + \nu(\nu+1)v = 0 \qquad (37)$$

In view of (6, b) with $\mu = 0$ we find that if we select $y = \sin x$, $u = v \cos^{1/2} x$ we can transform (37) to canonical form leading to solutions:

$$\phi_o(x,\lambda) = \cos^{1/2}x \cdot P_\nu(\sin x) \quad , \; \chi_o(x,\lambda) = \cos^{1/2} x$$
$$\cdot\, Q_\nu(\sin x) \quad , \qquad \lambda = \nu(\nu+1) \qquad (38)$$

where P_ν and Q_ν are Legendre functions of the first and second kind, respectively. In view of (14), when we apply the transform to (35) the second term gives $- \nu_n(\nu_n+1) - B_v(r,\lambda_n)$. Now B_v contains four arbitrary functions which must be specified on the surfaces $\theta = \theta_o$ and $\theta = \theta_1$. The terms containing the derivatives of v are not given. Thus if we select $\alpha = \beta = 0$ these terms drop out, leaving the terms containing $v_o(r)$ and $v_1(r)$, which are given. Hence $\phi(x,\lambda)$ and $\omega_o(\lambda)$ of (5) become:

$$\begin{aligned} \phi(x,\lambda) &= \cos^{1/2}x_o \cos^{1/2}x \, [P_\nu(\sin x)Q_\nu(\sin x_o) \\ &\quad - P_\nu(\sin x_o)Q_\nu(\sin x)] \, , \qquad x_o = \tfrac{\pi}{2} - \theta_o \, , \\ x_1 &= \tfrac{\pi}{2} - \theta_1 \, , \quad \omega_o(\lambda) = 1 \end{aligned} \tag{39}$$

By (11) and (8) we have:

$$\begin{aligned} \psi(x,\lambda_n) &= [k_n/\omega_{,\lambda}(\lambda_n)]^{1/2} \phi(x,\lambda_n) \, , \\ k_n &= \frac{\cos^{1/2}x_1 \, P_{\nu_n}(\sin x_1)}{\cos^{1/2}x_o \, P_{\nu_n}(\sin x_o)} \end{aligned} \tag{40}$$

Calculation of $\omega(\lambda) = \phi \chi_{,x} - \chi \phi_{,x}$ gives

$$\begin{aligned} \omega(\lambda) &= \cos^{1/2}x_o \cos^{1/2}x_1 \, [P_\nu(\sin x_o)Q_\nu(\sin x_1) \\ &\quad - P_\nu(\sin x_1)Q_\nu(\sin x_o)] \end{aligned} \tag{41}$$

Consequently, the roots $\lambda_n = \nu_n(\nu_n+1)$ of $\omega(\lambda) = 0$ satisfy the following equation

$$\frac{P_\nu(\sin x_1)}{P_\nu(\sin x_o)} = \frac{Q_\nu(\sin x_1)}{Q_\nu(\sin x_o)} \tag{42}$$

Therefore $\psi(x,\lambda_n)$ is completely determined.

After transforming (35) with $y = \sin x$ and $u = v \cos^{1/2}x$ we apply the finite Sturm-Liouville transform. That is, we multiply the equation by $\psi(x,\lambda_n)$ and integrate between x_o and x_1. The result is:

$$(r^2\bar{V}_{,r})_{,r} - \nu_n(\nu_n+1)\bar{V} = B_V(r,\lambda_n)$$
$$B_V(r,\lambda_n) = [k_n/\omega_{,\lambda}(\lambda_n)]^{1/2}[V_o(r) - k_n^{-1}V_1(r)] \tag{43}$$

Equation (43) is an ordinary differential equation of Euler type whose solution can be found by variation of parameters. Hence:

$$\bar{V}(r,\nu_n) = C_1(\nu_n)r^{\nu_n} + C_2(\nu_n)r^{-\nu_n-1} + F(r,\nu_n)$$
$$F(r,\nu_n) = \int^r \frac{B_V(\rho,\lambda_n)}{1+2\nu_n}(r^{\nu_n}\rho^{-\nu_n-1} - r^{-\nu_n-1}\rho^{\nu_n})\,d\rho \tag{44}$$

Let the transforms of $V_3(\theta)$ and $V_4(\theta)$ be $\bar{V}_3(\nu_n)$ and $\bar{V}_4(\nu_n)$. C_1 and C_2 will then be determined from the remaining conditions:

$$\begin{aligned} \bar{V} &= \bar{V}_3(\nu_n), \quad \text{for } r = r_o \\ \bar{V} &= \bar{V}_4(\nu_n), \quad \text{for } r = r_1 \end{aligned} \qquad \theta_o < \theta < \theta_1$$

This gives two linear equations for C_1 and C_2 whose solutions are:

$$C_1(\nu_n) = \frac{[\bar{V}_3(\nu_n) - F(r_o,\nu_n)]r_1^{-\nu_n-1} - [\bar{V}_4(\nu_n) - F(r_1,\nu_n)]r_o^{-\nu_n-1}}{r_o^{\nu_n}\,r_1^{-\nu_n-1} - r_1^{\nu_n}\,r_o^{-\nu_n-1}}$$
$$C_2(\nu_n) = \frac{[\bar{V}_4(\nu_n) - F(r_1,\nu_n)]r_o^{\nu_n} - [\bar{V}_3(\nu_n) - F(r_o,\nu_n)]r_1^{\nu_n}}{r_o^{\nu_n}\,r_1^{-\nu_n-1} - r_1^{\nu_n}\,r_o^{-\nu_n-1}} \tag{45}$$

Hence, $\bar{V}(r,\nu_n)$ is completely determined. The inversion theorem (12) now gives:

$$V(r,x) = \sum_n \bar{V}(r,\nu_n)\psi(x,\lambda_n) \tag{46}$$

where the summation is extended over all roots of (42).

The analysis given above is formal. It can, however, be made rigorous by showing that the solution satisfies both the differential equation and the boundary conditions.

It may be worth while to remark that with the use of the method of separation of variables the solution of the above problem would have been difficult and lengthy, if not impossible.

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Contributing Personnel:

Dr. A. C. Eringen, Research Associate Professor of Mechanics

Mr. T. King, Staff Computer

Respectfully submitted,

A. C. Eringen

A. C. Eringen

Wm R. Osgood

William R. Osgood
Chairman, Department
of Mechanics

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